

A symmetrization result for a class of anisotropic elliptic problems

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Abstract

We prove estimates for weak solutions to a class of Dirichlet problems associated to anisotropic elliptic equations with a zero order term..

1 Introduction

We consider the class of Dirichlet problems for anisotropic elliptic equations, whose prototype has the form

$$(1.1) \quad \begin{cases} -\sum_{i=1}^N \left(|u_{x_i}|^{p_i-2} u_{x_i} \right)_{x_i} + b(u) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of \mathbb{R}^N with Lipschitz continuous boundary, $N \geq 2$, $p_i \geq 1$ for $i = 1, \dots, N$ such that their harmonic mean \bar{p} is greater than 1, the subscript x_i denotes partial derivative with respect to x_i , b is a continuous, non-decreasing function such that $b(0) = 0$ and f is a nonnegative function with a suitable summability.

The anisotropy of problem (1.1) depends on differential operator whose growth with respect to the partial derivatives of u is governed by different powers. In the last years anisotropic problems have been extensively studied by many authors (see *e.g.* [AdBF2, AdBF3, ACh, BMS, DFG, DF, FGK, FGL, FS, G, Mar]).

The growing interest has led to an extensive investigation also for problems governed by fully anisotropic growth conditions (see [AC, A, AdBF1, C1, C3]) and problems related to different type of anisotropy (see *e.g.* [AFTL, BFK, DdB, DG]).

Our goal is to obtain an estimate of concentration of a weak solution to problem (1.1) via symmetrization methods. The use of the standard isoperimetric inequality in the study of isotropic elliptic Dirichlet problems was introduced in [Maz1, Maz2] and independently in [Ta1, Ta2]. Variants and extensions from these papers have been developed in a rich literature. We refer to Vazquez [V2] and Trombetti [T] for a quite comprehensive bibliography on this and related topics.

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It is well known that when isotropic elliptic Dirichlet problems with a zero order term are considered, the situation is quite different if we assume or not a sign condition (see, *e.g.*, [D1, D2, Mad, V1, V2]). In the anisotropic setting there are two different cases as well. Indeed, when $b(u)u \geq 0$, it is showed (see, *e.g.*, [C3]) that the symmetric rearrangement of a solution u to anisotropic problem (1.1) is pointwise dominated by the radial solution to an isotropic problem, defined in a ball, with a radially symmetric decreasing data and with no zero order term. Otherwise, with no sign condition on $b(u)u$, we prove an integral comparison result between a solution u to anisotropic problem (1.1) and the radial solution to a suitable isotropic problem defined in a ball, with a radially symmetric decreasing data again but, this time, which preserves a zero order term.

Just to give an idea of our results, let us consider problem (1.1) when the domain Ω is $B_R(0)$, the ball centered at the origin and with radius $R > 0$. We take into account two smooth strictly increasing functions b and \tilde{b} having the same domain such that $b(0) = \tilde{b}(0) = 0$, and two positive decreasing radial symmetric functions f and \tilde{f} defined in $B_R(0)$. Denote by b^{-1} and $(\tilde{b})^{-1}$ the inverse function of b and \tilde{b} , respectively. Suppose that

$$((\tilde{b})^{-1})'(s) \leq (b^{-1})'(s) \quad \text{for every } s \in \mathbb{R}$$

and that the datum f is less concentrated than the datum \tilde{f} , *i.e.*

$$\int_{B_r(0)} f(x) dx \leq \int_{B_r(0)} \tilde{f}(x) dx \quad \text{for every } 0 \leq r \leq R.$$

Then, we are going to prove that

$$\int_{B_r(0)} b(u^\star(x)) dx \leq \int_{B_r(0)} \tilde{b}(\tilde{u}(x)) dx \quad \text{for every } 0 \leq r \leq R,$$

where u^\star is the symmetric decreasing rearrangement of the solution u to problem (1.1) and \tilde{u} is the solution to the following problem

$$\begin{cases} -\operatorname{div}(|\nabla \tilde{u}|^{\bar{p}-2} \nabla \tilde{u}) + \tilde{b}(\tilde{u}) = \tilde{f}(x) & \text{in } B_R(0) \\ \tilde{u} = 0 & \text{on } \partial B_R(0). \end{cases}$$

The paper is organized as follows. In Section 2 we recall some backgrounds on the anisotropic spaces and on the properties of symmetrization. In Section 3 we state our main results, proved in Section 4.

2 Preliminaries

Let Ω be a bounded open subset of \mathbb{R}^N , $N \geq 2$, and let $1 \leq p_1, \dots, p_N < \infty$ be N real numbers. The anisotropic Sobolev space (see *e.g.* [Tr])

$$W^{1, \vec{p}}(\Omega) = \{u \in W^{1,1}(\Omega) : u_{x_i} \in L^{p_i}(\Omega), i = 1, \dots, N\}$$

is a Banach space with respect to the norm

$$(2.1) \quad \|u\|_{W^{1, \vec{p}}(\Omega)} = \sum_{i=1}^N \|u_{x_i}\|_{L^{p_i}(\Omega)}.$$

The space $W_0^{1, \vec{p}}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.1) and we will denote by $(W_0^{1, \vec{p}}(\Omega))'$ its dual.

A precise statement of our results requires the use of classical notions of rearrangement and of suitable symmetrization of a Young function, introduced by Klimov in [K].
Let u be a measurable function (continued by 0 outside its domain) fulfilling

$$(2.2) \quad |\{x \in \mathbb{R}^N : |u(x)| > t\}| < +\infty \quad \text{for every } t > 0.$$

The *symmetric decreasing rearrangement* of u is the function $u^\star : \mathbb{R}^N \rightarrow [0, +\infty[$ satisfying

$$(2.3) \quad \{x \in \mathbb{R}^N : u^\star(x) > t\} = \{x \in \mathbb{R}^N : |u(x)| > t\}^\star \quad \text{for } t > 0.$$

The *decreasing rearrangement* u^* of u is defined by

$$u^*(s) = \sup\{t > 0 : \mu_u(t) > s\} \quad \text{for } s \geq 0,$$

where

$$\mu_u(t) = |\{x \in \Omega : |u(x)| > t\}| \quad \text{for } t \geq 0$$

denotes the *distribution function* of u .

Moreover,

$$u^\star(x) = u^*(\omega_N |x|^N) \quad \text{for a.e. } x \in \mathbb{R}^N.$$

Similarly, we define the *symmetric increasing rearrangement* u_\star on replacing “ $>$ ” by “ $<$ ” in the definitions of the sets in (2.2) and (2.3). We refer to [BS] for details on these topics.

In this paper we will consider an N -dimensional Young function $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ (namely, an even convex function such that $\Phi(0) = 0$ and $\lim_{|\xi| \rightarrow +\infty} \Phi(\xi) = +\infty$) of the following type:

$$(2.4) \quad \Phi(\xi) = \sum_{i=1}^N \alpha_i |\xi_i|^{p_i} \quad \text{for } \xi \in \mathbb{R}^N \quad \text{with } \alpha_i > 0 \quad \text{for } i = 1, \dots, N.$$

We denote by $\Phi_\diamond : \mathbb{R} \rightarrow [0, +\infty[$ the symmetrization of Φ introduced in [K]. It is the one-dimensional Young function fulfilling

$$(2.5) \quad \Phi_\diamond(|\xi|) = \Phi_{\bullet\star\bullet}(\xi) \quad \text{for } \xi \in \mathbb{R}^N,$$

where Φ_\bullet is the Young conjugate function of Φ given by

$$\Phi_\bullet(\xi') = \sup \{ \xi \cdot \xi' - \Phi(\xi) : \xi \in \mathbb{R}^N \} \quad \text{for } \xi' \in \mathbb{R}^N.$$

So Φ_\diamond is the composition of Young conjugation, symmetric increasing rearrangement and Young conjugate again.

We denote by \bar{p} the *harmonic average* of the exponents p_i , i.e.

$$(2.6) \quad \frac{1}{\bar{p}} = \frac{1}{N} \sum_{i=1}^N \frac{1}{p_i}.$$

The *harmonic average* \bar{p} plays a basic role in discussing anisotropic equations of the form (1.1). Let us assume that $\bar{p} > 1$ and set

$$(2.7) \quad \Lambda = \frac{2^{\bar{p}} (\bar{p} - 1)^{\bar{p}-1}}{\bar{p}^{\bar{p}}} \left[\frac{\prod_{i=1}^N p_i^{\frac{1}{p_i}} (p'_i)^{\frac{1}{p'_i}} \Gamma(1 + 1/p'_i)}{\omega_N \Gamma(1 + N/\bar{p}')} \right]^{\frac{\bar{p}}{N}} \left(\prod_{i=1}^N \alpha_i^{\frac{1}{p_i}} \right)^{\frac{\bar{p}}{N}}$$

with ω_N the measure of the N -dimensional unit ball, Γ the Gamma function and $p'_i = \frac{p_i}{p_i-1}$, the Hölder conjugate of p_i with the usual conventions if $p_i = 1$. We are now in position to evaluate $\Phi_\diamond(|\xi|)$. Easy calculations show (see *e.g.* [C3]) that

$$(2.8) \quad \Phi_\diamond(|\xi|) = \Lambda |\xi|^{\bar{p}}.$$

In the anisotropic setting, we stress that \bar{p} plays a role also in a *Polya-Szegö principle* which reads as follows (see [C3]). Let u be a weakly differentiable function in \mathbb{R}^N satisfying (2.2) and such that

$$\sum_{i=1}^N \alpha_i \int_{\mathbb{R}^N} |u_{x_i}|^{p_i} dx < +\infty.$$

Then u^\star is weakly differentiable in \mathbb{R}^N and

$$(2.9) \quad \Lambda \int_{\mathbb{R}^N} |\nabla u^\star|^{\bar{p}} dx \leq \sum_{i=1}^N \alpha_i \int_{\mathbb{R}^N} |u_{x_i}|^{p_i} dx.$$

3 Main results

In the present section, we focus our attention on the following class of anisotropic elliptic problems

$$(3.1) \quad \begin{cases} -\operatorname{div}(a(x, u, \nabla u)) + g(x, u) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of \mathbb{R}^N with Lipschitz continuous boundary, $N \geq 2$, $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function such that for *a.e.* $x \in \Omega$, for all $s \in \mathbb{R}$ and for all $\xi, \xi' \in \mathbb{R}^N$

$$(A1) \quad a(x, s, \xi) \cdot \xi \geq \sum_{i=1}^N \alpha_i |\xi_i|^{p_i} \quad \text{with } \alpha_i > 0,$$

$$(A2) \quad |a_j(x, s, \xi)| \leq \beta \left[|s|^{\frac{\bar{p}}{p_j}} + |\xi_j|^{p_j-1} \right] \quad \text{with } \beta > 0 \quad \forall j = 1, \dots, N,$$

$$(A3) \quad (a(x, s, \xi) - a(x, s, \xi')) \cdot (\xi - \xi') > 0 \quad \text{for } \xi \neq \xi',$$

where $1 \leq p_1, \dots, p_N < \infty$ are real numbers and $\bar{p} > 1$.

Moreover, we assume that $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a measurable, continuous and non-decreasing function in s for fixed x , and bounded in x uniformly for bounded u such that

$$(A4) \quad g(x, s) s \geq b(s) s \quad \text{for a.e. } x \in \Omega, \forall s \in \mathbb{R}, \text{ where } b \text{ is a continuous and strictly increasing function such that } b(0) = 0.$$

Finally, we assume that

$$(A5) \quad f : \Omega \rightarrow \mathbb{R} \text{ is a nonnegative function such that } f \in \left(W_0^{1, \vec{p}}(\Omega) \right)'.$$

In order to give a precise statement of our results, we need to precise what means to be less diffusive. Let b_1, b_2 be two continuous strictly increasing functions. We say that b_1 is *weaker* than b_2 and we write

$$(3.2) \quad b_1 \prec b_2,$$

if they have the same domains and there exists a contraction¹ $\rho : \mathbb{R} \rightarrow \mathbb{R}$ such that $b_1 = \rho \circ b_2$.

We are interested in proving an integral estimate of a weak solution $u \in W_0^{1, \vec{p}}(\Omega)$ to problem (3.1) in terms of the weak solution $w \in W_0^{1, \vec{p}}(\Omega^\star)$ to the following problem

$$(3.3) \quad \begin{cases} -\operatorname{div}(\Lambda |\nabla w|^{\vec{p}-2} \nabla w) + \tilde{b}(w) = \tilde{f}(x) & \text{in } \Omega^\star \\ w = 0 & \text{on } \partial\Omega^\star, \end{cases}$$

where Ω^\star is the ball centered at the origin and having the same measure as Ω ,

(A6) \tilde{b} is a continuous and strictly increasing function such that $\tilde{b}(0) = 0$,

(A7) $(\tilde{b})^{-1} \prec b^{-1}$,

(A8) $\tilde{f} : \Omega^\star \rightarrow \mathbb{R}$ is a nonnegative radially symmetric function and decreasing along the radii such that $\tilde{f} \in \left(W_0^{1, \vec{p}}(\Omega^\star)\right)'$.

We stress that, by standard arguments and thanks to the results contained in [BB] (see also [BCE] for the anisotropic setting), there exists a unique weak solution $w \in W_0^{1, \vec{p}}(\Omega^\star)$ to (3.3) such that

$$(i) \quad \tilde{b}(w) \in L^1(\Omega^\star)$$

$$(ii) \quad \tilde{b}(w)w \in L^1(\Omega^\star)$$

$$(iii) \quad \Lambda \int_{\Omega} |\nabla w|^{\vec{p}-2} \nabla w \cdot \nabla \phi \, dx + \int_{\Omega} \tilde{b}(w) \phi \, dx = \langle \tilde{f}, \phi \rangle_{(W_0^{1, \vec{p}}(\Omega^\star))'}$$

for every $\phi \in W_0^{1, \vec{p}}(\Omega^\star) \cap L^\infty(\Omega^\star)$ and $\varphi = w$.

Theorem 3.1 *Assume that (A1)–(A8) hold. Let u be a weak solution to the problem (3.1) and w the weak solution to the problem (3.3). Then,*

$$(3.4) \quad \|(\mathcal{B} - \tilde{\mathcal{B}})_+\|_{L^\infty(0, |\Omega|)} \leq \|(\mathcal{F} - \tilde{\mathcal{F}})_+\|_{L^\infty(0, |\Omega|)},$$

where

$$(3.5) \quad \mathcal{B}(s) = \int_0^s b(u^*(t)) \, dt \quad \tilde{\mathcal{B}}(s) = \int_0^s \tilde{b}(w^*(t)) \, dt$$

$$(3.6) \quad \mathcal{F}(s) = \int_0^s f^*(t) \, dt \quad \tilde{\mathcal{F}}(s) = \int_0^s \tilde{f}^*(t) \, dt$$

for $s \in (0, |\Omega|]$.

If we assume that the datum of problem (3.1) dominates the datum of problem (3.3), then the following comparison result between concentrations holds as an easy consequence of Theorem 3.1.

¹By contraction we mean $|\rho(a) - \rho(b)| \leq |a - b|$ for $a, b \in \mathbb{R}$.

Corollary 3.2 *Under the same assumption of Theorem 3.1, if we suppose that*

$$(3.7) \quad \mathcal{F}(s) \leq \tilde{\mathcal{F}}(s) \quad \text{for any } s \in [0, |\Omega|],$$

then

$$(3.8) \quad \mathcal{B}(s) \leq \tilde{\mathcal{B}}(s) \quad \text{for any } s \in [0, |\Omega|].$$

In particular, we have

$$(3.9) \quad \int_{\Omega} \Psi(b(u(x))) \, dx \leq \int_{\Omega^{\star}} \Psi(\tilde{b}(w(x))) \, dx$$

for all convex and non-decreasing function $\Psi : \mathbb{R} \rightarrow \mathbb{R}$.

An immediate consequence of Corollary 3.2 are norm estimates of $b(u)$ in terms of norm of $\tilde{b}(w)$. An example of applications of (3.9) is the following one:

$$\|b(u)\|_{L^p(\Omega)} \leq \|\tilde{b}(w)\|_{L^p(\Omega^{\star})} \quad \text{for } 1 \leq p \leq \infty.$$

We emphasize that in the spirit of [V2], Theorem 3.1 and Corollary 3.2 still hold if we do not require the strictly monotony of b and \tilde{b} , but assume that b and \tilde{b} are non-decreasing functions or, more generally, maximal monotone graphs in \mathbb{R}^2 such that $b(0) \ni 0$ and $\tilde{b}(0) \ni 0$. Indeed, a maximal monotone graph is a natural generalization of the concept of monotone non-decreasing real function; moreover, the inverse of a maximal monotone graph is again a maximal monotone graph (see [V2] for more details).

4 Proof of Theorem 3.1

Let us consider the functions $u_{\kappa,t} : \Omega \rightarrow \mathbb{R}$ defined by

$$u_{\kappa,t}(x) = \begin{cases} 0 & \text{if } |u(x)| \leq t, \\ (|u(x)| - t) \operatorname{sign}(u(x)) & \text{if } t < |u(x)| \leq t + \kappa \\ \kappa \operatorname{sign}(u(x)) & \text{if } t + \kappa < |u(x)|, \end{cases}$$

for any fixed t and $\kappa > 0$. This function can be chosen as a test function in (3.1). By (A1) and (A4),

$$(4.1) \quad -\frac{d}{dt} \int_{\{|u|>t\}} \sum_{i=1}^N \alpha_i |u_{x_i}|^{p_i} \, dx \leq \int_{\{|u|>t\}} |f(x)| \, dx - \int_{\{|u|>t\}} b(u(x)) \operatorname{sign} u \, dx \quad \text{for a.e. } t > 0.$$

Taking into account (2.4), (2.8) and (2.9), analogous arguments as in [C3] yield

$$(4.2) \quad -\frac{d}{dt} \int_{\{u^{\star}>t\}} \Lambda |\nabla u^{\star}|^{\bar{p}} \, dx \leq -\frac{d}{dt} \int_{\{|u|>t\}} \sum_{i=1}^N \alpha_i |u_{x_i}|^{p_i} \, dx \quad \text{for a.e. } t > 0.$$

By the Coarea formula and the Hölder inequality,

$$(4.3) \quad \left(-\frac{d}{dt} \int_{\{u^{\star}>t\}} |\nabla u^{\star}|^{\bar{p}} \, dx \right)^{\frac{1}{\bar{p}}} \geq N \omega_N^{\frac{1}{N}} \mu_u(t)^{\frac{1}{N'}} (-\mu'_u(t))^{-\frac{1}{\bar{p}}} \quad \text{for a.e. } t > 0.$$

Since f is nonnegative, the maximum principle assures that $u \geq 0$. Since b is monotone, we obtain

$$(4.4) \quad \int_{\{|u|>t\}} b(u(x)) \operatorname{sign} u \, dx = \int_0^{\mu_u(t)} b(u^*(s)) \, ds \quad \text{for a.e. } t > 0.$$

Thus, as a consequence of (4.1), (4.2), (4.3) and (4.4), it follows that

$$(4.5) \quad \Lambda \left(N \omega_N^{\frac{1}{N}} \mu_u(t)^{\frac{1}{N'}} (-\mu'_u(t))^{-\frac{1}{\bar{p}'}} \right)^{\bar{p}} \leq \int_0^{\mu_u(t)} f^*(s) \, ds - \int_0^{\mu_u(t)} b(u^*(s)) \, ds \quad \text{for a.e. } t > 0.$$

The relation (4.5) implies that

$$(4.6) \quad 1 \leq \frac{-\mu'_u(t) \Lambda^{-\frac{1}{\bar{p}-1}}}{\left(N \omega_N^{\frac{1}{N}} \right)^{\frac{\bar{p}}{\bar{p}-1}} (\mu_u(t))^{\frac{\bar{p}'}{N'}}} [\mathcal{F}(\mu_u(t)) - \mathcal{B}(\mu_u(t))]^{\frac{1}{\bar{p}-1}} \quad \text{for a.e. } t > 0,$$

where \mathcal{F} and \mathcal{B} are defined as in (3.5) and (3.6), respectively.

By standard arguments (see, *e.g.*, [Ta1]), it follows that

$$(4.7) \quad (-u^*(s))' \leq \left(N \omega_N^{\frac{1}{N}} \right)^{-\bar{p}'} \Lambda^{-\frac{1}{\bar{p}-1}} s^{-\frac{\bar{p}'}{N'}} [\mathcal{F}(s) - \mathcal{B}(s)]^{\frac{1}{\bar{p}-1}} \quad \text{for a.e. } s \in (0, |\Omega|).$$

By (3.5),

$$(4.8) \quad \mathcal{B}'(s) = b(u^*(s)) \quad \text{for a.e. } s \in (0, |\Omega|).$$

Relations (3.5), (4.7) and (4.8) imply that

$$(4.9) \quad \begin{cases} \Lambda \left(N \omega_N^{\frac{1}{N}} \right)^{\bar{p}} s^{\frac{\bar{p}}{N'}} \left[-\frac{d}{ds} (\gamma(\mathcal{B}'(s))) \right]^{\bar{p}-1} + \mathcal{B}(s) \leq \mathcal{F}(s) & \text{for a.e. } s \in (0, |\Omega|) \\ \mathcal{B}(0) = 0, \quad \mathcal{B}'(|\Omega|) = 0, \end{cases}$$

where γ is the inverse function of b , *i.e.* $\gamma = b^{-1}$.

Let us consider problem (3.3). A weak solution w to problem (3.3) is unique and the symmetry of data assures that $w(x) = w(|x|)$, *i.e.* w is positive and radially symmetric. Moreover, setting $s = \omega_N |x|^N$ and $\tilde{w}(s) = w((s/\omega_N)^{1/N})$, we get that for all $s \in [0, |\Omega|]$

$$-\Lambda |\tilde{w}'(s)|^{\bar{p}-2} \tilde{w}'(s) = \frac{s^{-\bar{p}/N'}}{(N \omega_N^{1/N})^{\bar{p}}} \int_0^s (f^*(\sigma) - \tilde{b}(\tilde{w}(\sigma))) \, d\sigma \quad \text{for a.e. } s \in (0, |\Omega|).$$

Since it is possible to show (see [D1, Lemma 1.31]) that the above integral is positive, we deduce that $w(x) = w^\star(x)$. By the properties of w we can repeat arguments used to prove (4.7) replacing all the inequalities by equalities and obtaining

$$(4.10) \quad (-w^*(s))' = \left(N \omega_N^{\frac{1}{N}} \right)^{-\bar{p}'} \Lambda^{-\frac{1}{\bar{p}-1}} s^{-\frac{\bar{p}'}{N'}} [\tilde{\mathcal{F}}(s) - \tilde{\mathcal{B}}(s)]^{\frac{1}{\bar{p}-1}} \quad \text{for a.e. } s \in (0, |\Omega|).$$

Moreover, we have

$$(4.11) \quad \begin{cases} \Lambda \left(N \omega_N^{\frac{1}{N}} \right)^{\bar{p}} s^{\frac{\bar{p}}{N'}} \left[-\frac{d}{ds} (\tilde{\gamma}(\tilde{\mathcal{B}}'(s))) \right]^{\bar{p}-1} + \tilde{\mathcal{B}}(s) = \tilde{\mathcal{F}}(s) & \text{for a.e. } s \in (0, |\Omega|) \\ \tilde{\mathcal{B}}(0) = 0, \quad \tilde{\mathcal{B}}'(|\Omega|) = 0, \end{cases}$$

where $\tilde{\gamma}$ is the inverse function of \tilde{b} , i.e. $\tilde{\gamma} = (\tilde{b})^{-1}$.

Since $\mathcal{B}, \tilde{\mathcal{B}} \in \mathcal{C}([0, |\Omega|])$, there exists $s_0 \in (0, |\Omega|)$ such that

$$(4.12) \quad \|(\mathcal{B} - \tilde{\mathcal{B}})_+\|_{L^\infty(0, |\Omega|)} = (\mathcal{B} - \tilde{\mathcal{B}})(s_0).$$

In order to prove (4.7), we argue by contradiction. Assume that

$$(4.13) \quad (\mathcal{B} - \tilde{\mathcal{B}})(s_0) > \|(\mathcal{F} - \tilde{\mathcal{F}})_+\|_{L^\infty(0, |\Omega|)}.$$

We distinguish two cases: $s_0 < |\Omega|$ and $s_0 = |\Omega|$.

Case $s_0 < |\Omega|$. Combining (4.9) and (4.11) yields

$$(4.14) \quad \begin{aligned} & \Lambda \left(N \omega_N^{\frac{1}{N}} \right)^{\bar{p}} s^{\frac{\bar{p}}{N'}} \left[\left(-\frac{d}{ds} (\gamma(\mathcal{B}'(s))) \right)^{\bar{p}-1} - \left(-\frac{d}{ds} (\tilde{\gamma}(\tilde{\mathcal{B}}'(s))) \right)^{\bar{p}-1} \right] \\ & \leq \mathcal{F}(s) - \tilde{\mathcal{F}}(s) + \tilde{\mathcal{B}}(s) - \mathcal{B}(s) \quad \text{for a.e. } s \in (0, |\Omega|) \end{aligned}$$

By (4.13),

$$(4.15) \quad \mathcal{F}(s) - \tilde{\mathcal{F}}(s) + \tilde{\mathcal{B}}(s) - \mathcal{B}(s) \leq \|(\mathcal{F} - \tilde{\mathcal{F}})_+\|_{L^\infty(0, |\Omega|)} - (\mathcal{B} - \tilde{\mathcal{B}})(s) < 0$$

for $s \in (s_0 - \varepsilon, s_0 + \varepsilon)$. As a consequence of (4.14) and (4.15) we obtain

$$(4.16) \quad \begin{aligned} & \Lambda \left(N \omega_N^{\frac{1}{N}} \right)^{\bar{p}} s^{\frac{\bar{p}}{N'}} \left[\left(-\frac{d}{ds} (\gamma(\mathcal{B}'(s))) \right)^{\bar{p}-1} - \left(-\frac{d}{ds} (\tilde{\gamma}(\tilde{\mathcal{B}}'(s))) \right)^{\bar{p}-1} \right] \\ & = \Lambda \left(N \omega_N^{\frac{1}{N}} \right)^{\bar{p}} s^{\frac{\bar{p}}{N'}} \omega(s) \left[-\frac{d}{ds} (\gamma(\mathcal{B}'(s)) - \tilde{\gamma}(\tilde{\mathcal{B}}'(s))) \right] < 0, \end{aligned}$$

where

$$\omega(s) = (\bar{p} - 1) \int_0^1 \left\{ \left[\tau \left(-\frac{d}{ds} (\gamma(\mathcal{B}'(s))) \right) + (1 - \tau) \left(-\frac{d}{ds} (\tilde{\gamma}(\tilde{\mathcal{B}}'(s))) \right) \right]^{\bar{p}-2} \right\} d\tau > 0.$$

Setting

$$(4.17) \quad Z = \mathcal{B} - \tilde{\mathcal{B}} \in W^{2,\infty}(s_0 - \varepsilon, s_0 + \varepsilon),$$

we get

$$(4.18) \quad -\frac{d}{ds} (\tilde{\gamma}(\mathcal{B}'(s)) - \tilde{\gamma}(\tilde{\mathcal{B}}'(s))) = -\frac{d}{ds} (Z'(s) \eta(s)),$$

where

$$(4.19) \quad \eta(s) = \int_0^1 \tilde{\gamma}' \left(\tau \mathcal{B}'(s) + (1 - \tau) \tilde{\mathcal{B}}'(s) \right) d\tau > 0.$$

By (A7), we can conclude that

$$(4.20) \quad -\frac{d}{ds} (\gamma(\mathcal{B}'(s)) - \tilde{\gamma}(\mathcal{B}'(s))) \geq 0 \quad \text{for a.e. } s \in (0, |\Omega|).$$

Then, by (4.18) and (4.20),

$$(4.21) \quad -\frac{d}{ds} (Z'(s) \eta(s)) \leq -\frac{d}{ds} (\gamma(\mathcal{B}'(s)) - \tilde{\gamma}(\tilde{\mathcal{B}}'(s))) \quad \text{for a.e. } s \in (0, |\Omega|).$$

Finally, thanks to (4.16) and (4.21), we have

$$(4.22) \quad \Lambda \left(N \omega_N^{\frac{1}{N}} \right)^{\bar{p}} s^{\frac{\bar{p}}{N'}} \omega(s) \left(-\frac{d}{ds} (\eta(s) Z'(s)) \right) \leq \\ \leq \Lambda \left(N \omega_N^{\frac{1}{N}} \right)^{\bar{p}} s^{\frac{\bar{p}}{N'}} \omega(s) \left[-\frac{d}{ds} \left(\gamma(\mathcal{B}'(s)) - \tilde{\gamma}(\tilde{\mathcal{B}}'(s)) \right) \right] < 0 \quad \text{for a.e. } s \in (0, |\Omega|).$$

We can conclude that

$$(4.23) \quad -\frac{d}{ds} (\eta(s) Z'(s)) < 0 \quad \text{for } s \in (s_0 - \varepsilon, s_0 + \varepsilon),$$

which is in contradiction with the assumption (4.12), *i.e.* Z has a maximum in s_0 .

Case $s_0 = |\Omega|$. In this case, the inequality (4.23) holds for $s \in (|\Omega| - \varepsilon, |\Omega|)$. So $Z'(|\Omega|) > 0$, but this is not true since $Z'(|\Omega|) = 0$.

□

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References

- [A] A. Alberico, *Boundedness of solutions to anisotropic variational problems*, Comm. Part. Diff. Eq. **36** (2011), 470–486; Corrigendum, *ibid*, **41**, No. 5, 877–878 (2016).
- [AC] A. Alberico, A. Cianchi, *Comparison estimates in anisotropic variational problems*, Manuscripta Math. **126** (2008), 481–503.
- [AdBF1] A. Alberico, G. di Blasio, F. Feo, *A priori estimates for solutions to anisotropic elliptic problems via symmetrization*, Math. Nachr., Version of Record online : 25 OCT 2016, DOI: 10.1002/mana.201500282.
- [AdBF2] A. Alberico, G. di Blasio, F. Feo, *Estimates for solutions to anisotropic elliptic equations with zero order term*, Geometric Properties for Parabolic and Elliptic PDEs. Contributions of the 4th Italian-Japanese Workshop, GPPEPDEs, Palinuro, Italy, May 25–29, 2015, pp. 1–15, Springer (2016).
- [AdBF3] A. Alberico, G. di Blasio, F. Feo, *Comparison results for nonlinear anisotropic parabolic problems*, in press on Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl..
- [ALT] A. Alvino, G. Trombetti, P. L. Lions, *On optimization problems with prescribed rearrangements*, Nonlinear Anal. **13** (1989), 185–220.
- [AFTL] A. Alvino, V. Ferone, G. Trombetti, P. L. Lions, *Convex symmetrization and applications*, Ann. Inst. H. Poincaré Anal. Non Linéaire **14** (1997), 275–293.
- [ACh] S. Antontsev, M. Chipot, *Anisotropic equations: uniqueness and existence results*, Diff. Int. Eq. **21** (2008), 401–419.
- [BFK] M. Belloni, V. Ferone, B. Kawohl, *Isoperimetric inequalities, Wulff shape and related questions for strongly nonlinear elliptic equations*, Zeit. Angew. Math. Phys. **54** (2003), 771–789.

- [BCE] M. Bendahmane, M. Chrif, S. El Manouni, *An approximation result in generalized anisotropic Sobolev spaces and applications*, Z. Anal. Anwend. **30** (2011), 341–353.
- [BB] H. Brezis, F. E. Browder, *Some properties of higher order Sobolev spaces*, J. Math. Pures Appl. **61** (1982), 245–259.
- [BS] C. Bennett, R. Sharpley, *Interpolation of operators*, Pure and Applied Mathematics, 129, Academic Press, Inc., Boston, MA, 1988.
- [BMS] L. Boccardo, P. Marcellini, C. Sbordone, *L^∞ -regularity for variational problems with sharp nonstandard growth conditions*, Boll. Un. Mat. Ital. A **4** (1990), 219–225.
- [C1] A. Cianchi, *Local boundedness of minimizers of anisotropic functionals*, Ann. Inst. Henri Poincaré, Analyse non linéaire **17** (2000), 147–168.
- [C3] A. Cianchi, *Symmetrization in anisotropic elliptic problems*, Comm. Part. Diff. Eq. **32** (2007), 693–717.
- [DdB] F. Della Pietra, G. di Blasio, *Blow-up solutions for some nonlinear elliptic equations involving a Finsler-Laplacian*, Publ. Mat. **61**, No. 1, 213–238 (2017).
- [DG] F. Della Pietra, N. Gavitone, *Anisotropic elliptic equations with general growth in the gradient and Hardy-type potentials*, J. Differential Equations **255** (2013), 3788–3810.
- [DFG] R. Di Nardo, F. Feo, O. Guibé, *Uniqueness result for nonlinear anisotropic elliptic equations*, Adv. Differential Equations **18** (2013), 433–458.
- [DF] R. Di Nardo, F. Feo, *Existence and uniqueness for nonlinear anisotropic elliptic equations*, Arch. Math. (Basel) **102** (2014), 141–153.
- [D1] J. I. Díaz, *Nonlinear partial differential equations and free boundaries. Vol. I. Elliptic equations*, Research Notes in Mathematics, **106**. Pitman, Boston, MA, 1985.
- [D2] J. I. Díaz, *Inequalities of isoperimetric type for the Plateau problem and the capillarity problem*, Rev. Acad. Canaria Cienc. **3** (1991), 127–166.
- [FGK] I. Fragalà, F. Gazzola, B. Kawohl, *Existence and nonexistence results for anisotropic quasilinear elliptic equations*, Ann. Inst. Henri Poincaré, Analyse non linéaire **21** (2004), 715–734.
- [FGL] I. Fragalà, F. Gazzola, G. Lieberman, *Regularity and nonexistence results for anisotropic quasilinear elliptic equations in convex domains*, Disc. Cont. Dynam. Syst. (2005), 280–286.
- [FS] N. Fusco, C. Sbordone, *Some remarks on the regularity of minima of anisotropic integrals*, Comm. Part. Diff. Equat. **18** (1993), 153–167.
- [G] M. Giaquinta, *Growth conditions and regularity, a counterexample*, Manus. Math. **59** (1987), 245–248.
- [Mad] C. Maderna, *Optimal problems for a certain class of nonlinear Dirichlet problems*, Boll. Un. Mat. Ital. Suppl. **1** (1980), 31–43.
- [Mar] P. Marcellini, *Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions*, Arch. Rat. Mech. Anal. **105** (1989), 267–284.
- [Maz1] V. G. Maz'ya, *Some estimates of solutions of second-order elliptic equations*, Dokl. Akad. Nauk. SSSR **137** (1961), 1057–1059 (Russian); English translation: Soviet Math. Dokl. **2** (1961), 413–415.

- [Maz2] V. G. Maz'ya, *On weak solutions of the Dirichlet and Neumann problems*, Trudy Moskov. Mat. Obšč. **20** (1969), 137–172 (Russian); English translation: Trans. Moscow Math. Soc. **20** (1969), 135–172.
- [K] V. S. Klimov, *Isoperimetric inequalities and imbedding theorems*, (Russian) Dokl. Akad. Nauk SSSR **217** (1974), 272–275.
- [Ta1] G. Talenti, *Elliptic equations and rearrangements*, Ann. Sc. Norm. Sup. Pisa IV **3** (1976), 697–718.
- [Ta2] G. Talenti, *Nonlinear elliptic equations, rearrangements of functions and Orlicz spaces*, Ann. Mat. Pura Appl. **120** (1979), 160–184.
- [Tr] M. Troisi, *Teoremi di inclusione per spazi di Sobolev non isotropi*, Ricerche Mat. **18** (1969), 3–24.
- [T] G. Trombetti, *Symmetrization methods for partial differential equations*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. **3** (2000), 601–634.
- [V1] J. L. Vazquez, *Symétrisation pour $u_t = \Delta\varphi(u)$ et applications*, C. R. Acad. Sci. Paris Sér. I Math. **295** (1982), 71–74.
- [V2] J. L. Vazquez, *Symmetrization and Mass Comparison for Degenerate Nonlinear Parabolic and related Elliptic Equations*, Advances in Nonlinear Studies, **5** (2005), 87–131.